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VELOCITY FIELD EQUATIONS AND STRAIN LOCALIZATION†

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Abstract—The velocity field equations of a hypo-elastic material are derived using arbitrary curvilinear coordinates in the actual configuration of the body, and emphasizing "higher order" terms which would disappear in an infinitesimal strain theory. As an example, bifurcation analyses are carried out with the help of the velocity field equations, for the thin rod and the thin plate under uniaxial tension. The results agree with those from the literature, at least if the "higher order" terms are disregarded. Differences may result from the way these terms are incorporated in the hypo-elastic approach.

1. INTRODUCTION

In Section 2 of this paper the velocity field equations will be derived for the actual configuration of a hypo-elastic body, using arbitrary (curvilinear) coordinates. They correspond to the Navier–Stokes equations of a viscous fluid, or to the Navier equations of linear elasticity, and govern the instantaneous velocity field inside the material provided that corresponding boundary conditions are prescribed. In the classical way the velocity field equations will be obtained by substituting the hypo-elastic constitutive law, i.e.

$$\dot{\sigma}^j{}_k = L^j{}_k{}^m{}_n \lambda_m{}^n \quad \text{where} \quad L^j{}_k{}^m{}_n = L^j{}_{kn}{}^m \tag{1}$$

into the equilibrium conditions

$$\sigma^j{}_{k|j} + \rho b_k = 0. \tag{2}$$

Both equations are written, with respect to the subsequent analysis in a mixed variant form. The stresses, $\sigma^j{}_k$ and the time derivative, will be discussed below. The strain rates are denoted by $\lambda_m{}^n$, i.e.

$$\lambda_{mn} = \frac{1}{2}(v_{m|n} + v_{n|m}) = \lambda_{nm} \tag{3}$$

where v_m are the point velocities, and $|_n$ represents a covariant derivative with respect to the point coordinates, ξ^n. Body forces are denoted by b_k, and ρ is the local material density. The coefficients, $L^j{}_k{}^m{}_n$ depend on the material considered, and on the mode of the time derivative chosen.

If as an example, the coordinates ξ^k define a fixed reference configuration, of the body under consideration then in a simple way

$$\dot{\sigma}^j{}_k = \frac{\partial}{\partial t}\sigma^j{}_k(\xi, t) \tag{4}$$

might be chosen as the partial time derivative so that after differentiating eqn (2), and regarding eqns (1) and (3) the velocity field equations take the form

† Dedicated to E. Pestel, Hannover, on the occasion of his 70th birthday on 29 May 1984.

$$L^j{}_k{}^m{}_n v^n|_{mj} = -(L^j{}_k{}^m{}_n|_j v^n|_m + [\rho b_k]^{\cdot}). \tag{5}$$

It was given in a more special shape, i.e. with the right-hand side to vanishing, by Nemat-Nasser[1] who explicitly pointed out their fundamental meaning. However, it has to be noticed that in order to ensure the validity of the equilibrium equations, eqn (2), the non-symmetric Piola stress should be used so that the coefficients **L** also become non-symmetric with respect to j and k.

This difficulty does not exist in the actual configuration of the body to which we will confine our attention from now on. Then $\sigma^j{}_k$ represents the symmetric Cauchy stress which reduces, in Cartesian coordinates to force per unit area. Lagrangian convected point coordinates are denoted ξ^k. Therefore $\sigma^j{}_k$ refers to local axes which change in time. If we again apply eqn (4) we now obtain the so-called Oldroyd time derivative. It must no longer by permutated with the spatial derivatives. This will result in "higher order" terms, which appear additionally in the coefficients of the corresponding velocity field equations to be derived in Section 2, and also will force us to carefully re-examine in Section 3 the form of the coefficients $L^j{}_k{}^m{}_n$ of the hypo-elastic law itself, with respect to terms having for an elastic–plastic body the same order of magnitude.

Finally in Section 4 the use of the velocity field equations will be demonstrated for the example of a bifurcation analysis, for the generalized one-dimensional continuum (thin rod) as well as for the generalized two-dimensional continuum (thin plate), both under uniaxial tension.

2. VELOCITY FIELD EQUATIONS IN THE ACTUAL CONFIGURATION OF THE BODY

As already mentioned in Section 1 we denote by $\dot{\sigma}^j{}_k$ the so-called Oldroyd time derivative of Cauchy stress. It might easily be transformed into the more frequently used Jaumann derivative, $\overset{\nabla}{\sigma}{}^j{}_k$ which refers to rigidly rotating local axes by

$$\overset{\nabla}{\sigma}{}^j{}_k - \dot{\sigma}^j{}_k = \lambda^j{}_p \sigma^p{}_k - \lambda^p{}_k \sigma^j{}_p \tag{6}$$

(cf. Ref. [12]). However, the Oldroyd derivative has the advantage that it leads from some invariant measures of strain ε directly to the strain rate λ. For instance if g_{jk}, g^{jk} are the covariant or contravariant components, respectively, of the metric unit tensor, and $g_{j_0k_0}$, $g^{j_0k_0}$ are the corresponding components referring to the initial (i.e. the reference) configuration of the body then we have

$$\frac{1}{2}\dot{g}_{jk} = \lambda_{jk}, \qquad \frac{1}{2}\dot{g}^{jk} = -\lambda^{jk}, \qquad \dot{\varepsilon}_{jk} = \lambda_{jk} \tag{7}$$

(cf. Ref. [2]). Here

$$\varepsilon_{jk} = \frac{1}{2}(g_{jk} - g_{j_0k_0})$$

is the Almansi strain in the actual configuration. Besides eqns (1) and (3) the symmetry relations

$$\varepsilon_j{}^k = \varepsilon^k{}_j, \qquad \sigma^j{}_k = \sigma_k{}^j, \qquad L^j{}_k{}^m{}_n = L_k{}^{jm}{}_n \tag{8}$$

are assumed to hold.

Denoting by $\Gamma^j{}_{kl}$ the Christoffel symbols, and by$,_j$ the partial derivative with respect to the (moving Lagrangian) coordinate ξ^j we form the covariant derivative in the usual way, i.e.

$$\sigma^l{}_k|_j = \sigma^l{}_{k,j} + \sigma^p{}_k\Gamma^l{}_{pj} - \sigma^l{}_p\Gamma^p{}_{kj}.$$

Differentiating this expression according to eqn (4) with respect to time, and forming $\dot{\sigma}^l{}_k|_j$ correspondingly yields

$$(\sigma^l{}_k|_j)^{\cdot} = (\sigma^l{}_{k'j})^{\cdot} + (\sigma^p{}_k\Gamma^l{}_{pj})^{\cdot} - (\sigma^l{}_p\Gamma^p{}_{kj})^{\cdot}$$
$$\dot{\sigma}^l{}_k|_j = \dot{\sigma}^l{}_{k'j} + \dot{\sigma}^p{}_k\Gamma^l{}_{pj} - \dot{\sigma}^l{}_p\Gamma^p{}_{kj}.$$

Therefrom

$$(\sigma^l{}_k|_j)^{\cdot} - \dot{\sigma}^l{}_k|_j = \sigma^p{}_k\dot{\Gamma}^l{}_{pj} - \sigma^l{}_p\dot{\Gamma}^p{}_{kj} \tag{9}$$

is obtained. On the other hand, as the total deformation represents a compatible process in the uncurved Euclidean space the Christoffel symbols obey the relations

$$\Gamma^k{}_{jp} = \frac{1}{2}g^{kr}[-g_{jp'r} + g_{rj'p} + g_{pr'j}]. \tag{10}$$

Differentiating them with respect to time, and observing eqns (7) gives us

$$\dot{\Gamma}^k{}_{jp} = -\lambda^{kr}[-g_{jp'r} + g_{rj'p} + g_{pr'j}]$$
$$+ g^{kr}[-\lambda_{jp'r} + \lambda_{rj'p} + \lambda_{pr'j}].$$

Therefore, we obtain regarding eqn (10), i.e. $-g_{jp'r} + g_{rj'p} + g_{pr,j} = 2g_{rl}\Gamma^l{}_{jp}$ that

$$\dot{\Gamma}^k{}_{jp} = g^{kr}\{-2\lambda_{rl}\Gamma^l{}_{jp} - \lambda_{jp'r} + \lambda_{rj'p} + \lambda_{pr'j}\}.$$

We want to substitute this expression observing eqn (3), into the right-hand side of eqn (9). This we do especially in Cartesian coordinates for which, $\Gamma^l{}_{jp} = 0$, and the covariant derivative coincides with the partial one. In this way

$$(\sigma^l{}_k|_j)^{\cdot} - \dot{\sigma}^l{}_k|_j = \sigma^p{}_k v^l|_{pj} - \sigma^l{}_p v^p|_{kj}$$

arises. As this is a tensorial relation it also holds automatically in any arbitrary curvilinear coordinates. So we may for $j = l$ substitute into the left-hand side the constitutive law, eqn (1), having the form

$$\dot{\sigma}^j{}_k|_j = L^j{}_k{}^m{}_n|_j\lambda_m{}^n + L^j{}_k{}^m{}_n\lambda_m{}^n|_j$$

as well as the equilibrium conditions, eqn (2), i.e.

$$(\sigma^j{}_k|_j)^{\cdot} = -\rho\left(\dot{b}_k + \frac{\dot{\rho}}{\rho}b_k\right).$$

Thereby we obtain because of eqn (3), $\dot{\rho}/\rho = -\lambda_l{}^l$, and eqn (1) the velocity field equations searched for, i.e.

$$\bar{L}^j{}_k{}^m{}_n v^n|_{mj} = -\rho\dot{b}_k + (\rho g^m{}_n b_k - L^j{}_k{}^m{}_n|_j)v^n|_m \tag{11}$$

the coefficients of which are given by

$$\bar{L}^j{}_k{}^m{}_n = L^j{}_k{}^m{}_n + g^j{}_n\sigma^m{}_k - g^m{}_k\sigma^j{}_n. \tag{12}$$

Because of eqn (6) the coefficients $\bar{L}^j{}_k{}^m{}_n$ could replace $L^j{}_k{}^m{}_n$ in the constitutive law, eqn (1), provided the Jaumann derivative is used. However, in contrast to, $L^j{}_k{}^m{}_n$ the coefficients, $\bar{L}^j{}_k{}^m{}_n$ need not automatically fulfil the symmetry relations given in eqn (1). Thus they are not uniquely determined, and should be taken from eqn (12).

3. ELASTIC–PLASTIC MATERIAL

From the very beginning we want to include, besides the $\mu = 3$ dimensional elastic–plastic continuum, also the generalized $\mu = 2$ dimensional one (thin plate, thickness H), and the generalized $\mu = 1$ dimensional one (thin rod, cross-section A). Therefore, we introduce the factor

$$w(\mu) = \begin{cases} 1 & \text{if } \mu = 3 \\ H & \text{if } \mu = 2 \\ A & \text{if } \mu = 1 \end{cases}$$

and now denote by $\bar{\sigma}^j{}_k$ the true (three-dimensional) Cauchy stress, by $\bar{Y}$ the true uniaxial yield limit, by $\bar{\rho}$ the true three-dimensional mass density, and by $\mathring{\bar{\rho}}$ its value in the initial configuration. Then the corresponding generalized quantities for 1, 2 or 3 dimensions become, respectively

$$\sigma^j{}_k = w\bar{\sigma}^j{}_k, \qquad Y = w\bar{Y}, \qquad \rho = w\bar{\rho}/\mathring{\bar{\rho}}. \tag{13}$$

Let us start with a so-called purely hyper-elastic deformation, $\varepsilon = \overset{E}{\varepsilon}$ for which the constitutive equation reads

$$\sigma^j{}_k = \rho\partial\phi/\partial\overset{E}{\varepsilon}{}_j{}^k \tag{14}$$

(cf. Ref. [3]). The time independent elastic potential obeying, $\phi(0) = 0$ is $\phi = \phi(\overset{E}{\varepsilon}{}_j{}^k)$. While a material fulfilling eqn (14) is truely elastic in a sense that the internal work becomes independent of the strain path, this property need no longer hold for materials containing w rather than ρ in eqn (14). Unfortunately just these ones are generally called "elastic". Because of $\dot{\rho}/\rho = -\lambda_l{}^l$, $\overset{E}{\varepsilon}{}_m{}^n = g^{np}\overset{E}{\varepsilon}{}_{mp}$, eqn (7), and after choosing ϕ in a way that the quantities

$$\phi^j{}_k{}^m{}_n = \partial^2\phi/\partial\overset{E}{\varepsilon}{}_j{}^k\partial\overset{E}{\varepsilon}{}_m{}^n = \phi^m{}_n{}^j{}_k \tag{15}$$

also fulfil the symmetry conditions given for **L** in eqn (1) it follows from differentiating eqn (14) according to eqn (4) that

$$\dot{\sigma}^j{}_k = \rho\{-\lambda_l{}^l\partial\phi/\partial\overset{E}{\varepsilon}{}_j{}^k + \phi^j{}_k{}^m{}_m(-2\lambda^{np}\overset{E}{\varepsilon}{}_{mp} + g^{np}\lambda_{mp})\}.$$

The first term inside the brackets becomes different for an "elastic" material, and then vanishes if $\mu = 3$. Now we denote the coefficients of the hyper-elastic constitutive law, eqn (1) by $E^j{}_k{}^m{}_n$ and obtain regarding eqn (14)

$$E^j{}_k{}^m{}_n = -g^m{}_n\sigma^j{}_k + \rho(g^m{}_p - 2\overset{E}{\varepsilon}{}^m{}_p)\phi^j{}_k{}^p{}_n.$$

These expressions are symmetric with respect to j and k. They also should be symmetrized by means of the substitution, $E^j{}_k{}^m{}_n \rightarrow (1/2)(E^j{}_k{}^m{}_n + E^j{}_{kn}{}^m)$ with respect to m and n. This gives us because of $E^j{}_{kn}{}^m = g_{nr}g^{ms}E^j{}_k{}^r{}_s$ for the hyper-elastic body

$$E^j{}_k{}^m{}_n = -\sigma^j{}_k g^m{}_n + \frac{\rho}{2}[(g^m{}_p - 2\overset{E}{\varepsilon}{}^m{}_p)\phi^j{}_k{}^p{}_n + (g^p{}_n - 2\overset{E}{\varepsilon}{}^p{}_n)\phi^j{}_k{}^m{}_p]$$

$$= -\sigma^j{}_k g^m{}_n - \rho[\overset{E}{\varepsilon}{}^m{}_p\phi^j{}_k{}^p{}_n + \overset{E}{\varepsilon}{}^p{}_n\phi^j{}_k{}^m{}_p] + \rho\phi^j{}_k{}^m{}_n \qquad (16)$$

while for an "elastic" body if $\mu = 3$

$$E^j{}_k{}^m{}_n = \frac{1}{2}(g^m{}_p - 2\overset{E}{\varepsilon}{}^m{}_p)\phi^j{}_k{}^p{}_n + \frac{1}{2}(g^p{}_n - 2\overset{E}{\varepsilon}{}^p{}_n)\phi^j{}_k{}^m{}_p$$

$$= \phi^j{}_k{}^m{}_n - [\overset{E}{\varepsilon}{}^m{}_p\phi^j{}_k{}^p{}_n + \overset{E}{\varepsilon}{}^p{}_n\phi^j{}_k{}^m{}_p]$$

holds. Should the Jaumann derivative be used rather than the Oldroyd derivative in eqn (1), we would obtain after adding eqn (6) to eqn (1) the substitution

$$E^j{}_k{}^m{}_n \rightarrow E^j{}_k{}^m{}_n + g^j{}_n\sigma^m{}_k - g^m{}_k\sigma^j{}_n. \qquad (17)$$

In the event of so-called linear, isotropic elasticity the function ϕ is usually chosen according to

$$\phi = \frac{1}{2}\phi^j{}_k{}^m{}_n\overset{E}{\varepsilon}{}_j{}^k\overset{E}{\varepsilon}{}_m{}^n = \frac{E}{2(1+\nu)}\left[\overset{E}{\varepsilon}{}_m{}^k\overset{E}{\varepsilon}{}_k{}^m + \frac{\nu}{1-(\mu-1)\nu}\overset{E}{\varepsilon}{}_j{}^j\overset{E}{\varepsilon}{}_k{}^k\right]$$

where ν denotes the Poisson number and, E Young's modulus. Observing the assumed symmetry in m and n, as well as in j and k it follows that

$$\phi^j{}_k{}^m{}_n = \frac{E}{1+\nu}\left[\frac{1}{2}(g^j{}_n g^m{}_k + g^{jm}g_{nk}) + \frac{\nu}{1-(\mu-1)\nu}g^j{}_k g^m{}_n\right]. \qquad (18)$$

Though these coefficients may via the quantities g^{jm} and g_{nk} depend on time t the function ϕ itself does not.

Notice that $E^j{}_k{}^m{}_n$ is in general, not identical with $\phi^j{}_k{}^m{}_n$ even if in eqn (1), the Jaumann derivative would be introduced rather than the Oldroyd one. Of course the difference terms are "higher order" ones.

In contrast to $E^j{}_k{}^m{}_n$ the expressions given in the literature (cf. Ref. [4]) for the coefficients in eqn (1) of an elastic–plastic rate-independent material, i.e.

$$L^j{}_k{}^m{}_n = E^j{}_k{}^m{}_n - \frac{E^j{}_k{}^p{}_q\dfrac{\partial g}{\partial\sigma^p{}_q}E^r{}_s{}^m{}_n\dfrac{\partial f}{\partial\sigma^r{}_s}}{h + E^r{}_s{}^p{}_q\dfrac{\partial f}{\partial\sigma^r{}_s}\dfrac{\partial g}{\partial\sigma^p{}_q}} \qquad (19)$$

would be correct, provided the quantities $E^j{}_k{}^m{}_n$ are correctly substituted as given above, and the Oldroyd time derivative is used rather than the Jaumann derivative, or any other one. The latter statement is due to the fact that in the usual proof of eqn (19) just the Oldroyd, i.e. the partial time derivative, eqn (4), is being applied. The hardening parameter h is defined by

$$h = \frac{\mathrm{d}Y}{\mathrm{d}\bar{\varepsilon}} \tag{20}$$

while the yield condition is assumed to read $f(\boldsymbol{\sigma}) = Y(\bar{\varepsilon})$ where f represents an appropriate, time-independent function of stress. Besides, a flow potential $g(\boldsymbol{\sigma})$ is introduced, which is identical with f provided the so-called normality rule holds (standard plasticity). Equation (19) is also based on the assumptions that the elastic and plastic strain rate may be superimposed to yield the total strain rate. The equivalent strain is denoted by $\bar{\varepsilon}$. It is defined by $\dot{\bar{\varepsilon}} = \bar{\lambda}$ with the equivalent strain rate $\bar{\lambda}$ given by

$$\bar{\lambda} = \frac{E^{j}{}_{k}{}^{m}{}_{n}\dfrac{\partial f}{\partial \sigma^{j}{}_{k}}\lambda_{m}{}^{n}}{h + E^{p}{}_{q}{}^{r}{}_{s}\dfrac{\partial f}{\partial \sigma^{p}{}_{q}}\dfrac{\partial g}{\partial \sigma^{r}{}_{s}}} \geq 0$$

($\bar{\lambda} < 0$ means elastic unloading).

In view of eqn (13) the functions f, g should be homogeneous, i.e. $f(\beta\boldsymbol{\sigma}) = |\beta| f(\boldsymbol{\sigma})$ and $g(\beta\boldsymbol{\sigma}) = |\beta| g(\boldsymbol{\sigma})$ for any scalar multiplier β.

Let us deal with two examples in the event of standard plasticity, i.e. $f \equiv g$. First we consider the thin uniaxial rod ($\mu = 1$). For it the yield condition becomes $f(\boldsymbol{\sigma}) = |\sigma^{1}{}_{1}| = Y$, i.e. $\partial f/\partial \sigma^{1}{}_{1} = \operatorname{sgn}(\sigma^{1}{}_{1})$. Using Cartesian coordinates, i.e. $g^{1}{}_{1} = 1$ and linear hyper-elasticity, i.e.

$$\overset{E}{\varepsilon}{}^{1}{}_{1} = \frac{\sigma^{1}{}_{1}}{\rho E}$$

eqns (16), (18) and (19) yield

$$\left.\begin{aligned} L^{1}{}_{1}{}^{1}{}_{1} = \frac{h}{1 + h/E^{1}{}_{1}{}^{1}{}_{1}}, \quad & E^{1}{}_{1}{}^{1}{}_{1} = \rho E - 3\sigma^{1}{}_{1}, \\ & |\sigma^{1}{}_{1}| = Y. \end{aligned}\right\} \tag{21}$$

As a second example we examine the homogeneous, thin isotropic plate ($\mu = 2$, thickness H) as described by the Cartesian coordinates ξ^1 (longitudinal), ξ^2 (transversal), and ξ^3 (orthogonal to the plate). It is loaded by a longitudinal traction $\sigma^{1}{}_{1}$ only, i.e. $\sigma^{1}{}_{2} = \sigma^{2}{}_{1} = 0$ and

$$\sigma^{1}{}_{1} = Y > 0, \qquad \sigma^{2}{}_{2} = 0, \qquad \sigma^{3}{}_{3} = 0. \tag{22}$$

Then the Tresca yield condition gives us

$$f(\sigma^{j}{}_{k}) = \sigma^{1}{}_{1} - \sigma^{2}{}_{2} = \sigma^{1}{}_{1} - \sigma^{3}{}_{3} = Y.$$

If only the in-plane deformation of the plate is considered then

$$\frac{\partial f}{\partial \sigma^{1}{}_{1}} = 1, \qquad \frac{\partial f}{\partial \sigma^{2}{}_{2}} = -1$$

represents the only non-vanishing components of $\partial f/\partial \sigma^{j}{}_{k}$.

Regarding instead the deformation in an orthogonal plane we obtain correspondingly

$$\frac{\partial f}{\partial \sigma^1{}_1} = 1, \qquad \frac{\partial f}{\partial \sigma^3{}_3} = -1.$$

Now we superpose both modes at the considered edge regime of the Tresca yield surface (cf. Ref. [5]). Using factors κ or $1-\kappa$, respectively, with

$$0 \le \kappa \le 1 \tag{23}$$

gives us

$$\frac{\partial f}{\partial \sigma^1{}_1} = 1, \qquad \frac{\partial f}{\partial \sigma^2{}_2} = -\kappa \tag{24}$$

while $\partial f/\partial \sigma^3{}_3 = -\kappa$ need not be considered any longer. In a similar way the Huber–v.-Mises yield condition, i.e.

$$f = \sqrt{\left\{\frac{1}{2}[(\sigma^1{}_1 - \sigma^2{}_2)^2 + (\sigma^2{}_2 - \sigma^3{}_3)^2 + (\sigma^3{}_3 - \sigma^1{}_1)^2]\right\}} = Y$$

gives under conditions (20) that

$$\frac{\partial f}{\partial \sigma^1{}_1} = 1, \qquad \frac{\partial f}{\partial \sigma^2{}_2} = -\frac{1}{2}.$$

Therefore eqns (23) and (24) are correct in all cases if κ could be chosen appropriately, i.e.

$$\left.\begin{array}{ll} \kappa = 1 & \text{for in-plane deformation, Tresca} \\ \kappa = 0 & \text{for orthogonal deformation, Tresca} \\ \kappa = 1/2 & \text{for general deformation, Huber–v.-Mises.} \end{array}\right\} \tag{25}$$

Resolving for a linear hyper-elastic material eqn (14) and observing eqns (15) and (18) leads to

$$\overset{E}{\varepsilon}_1{}^1 = \frac{1}{\rho E}\sigma^1{}_1 = \frac{Y}{\rho E}$$

$$\overset{E}{\varepsilon}_2{}^2 = -\frac{1}{\rho E}\nu\sigma^1{}_1 = -\frac{Y}{\rho E}\nu$$

$$\overset{E}{\varepsilon}_1{}^2 = \frac{1+\nu}{\rho E}\sigma^2{}_1 = 0.$$

These equations have to be substituted into eqn (16). We introduce the notations

$$\left.\begin{array}{ll} \eta = \dfrac{Y}{\rho E}, & \bar{h} = \dfrac{1+\nu}{2\rho E}h = \dfrac{1+\nu}{2\rho E}\dfrac{\mathrm{d}Y}{\mathrm{d}\bar{\varepsilon}} \\ L^j{}_k{}^m{}_n = \dfrac{\rho E}{1+\nu}M^j{}_k{}^m{}_n & \end{array}\right\} \tag{26}$$

and observe that "higher order" terms may obviously be identified by the factor η. Confining our attention to instantaneous Cartesian coordinates only, i.e.

$$g^{jk} = g_{jk} = g_j^{\ k} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases} \tag{27}$$

we obtain after some elementary though elaborate calculations from eqns (16), (18) and (19) regarding eqn (24)

$$\left.\begin{aligned}
P = M^1{}_1{}^1{}_1 &= \frac{1 - \eta(3 - v^2)}{1 - v} - \frac{1}{2\gamma(1 - v)^2}\{1 - \eta(3 - v^2) \\
&\quad - \kappa[v + \eta(3v^2 - 1)]\}\{1 - \eta(3 - v^2) - \kappa v(1 - 2\eta)\} \\
Q = M^2{}_2{}^2{}_2 &= \frac{1 + 2v\eta}{1 - v} \\
&\quad - \frac{1}{2\gamma(1 - v)^2}\{v(1 - 2\eta) - \kappa(1 + 2v\eta)\}\{v + \eta(3v^2 - 1) - \kappa(1 + 2v\eta)\} \\
U = M^1{}_1{}^2{}_2 &= \frac{v + \eta(3v^2 - 1)}{1 - v} - \frac{1}{2\gamma(1 - v)^2}\{1 - \eta(3 - v^2) \\
&\quad - \kappa[v + \eta(3v^2 - 1)]\}\{v + \eta(3v^2 - 1) - \kappa(1 + 2v\eta)\} \\
W = M^2{}_2{}^1{}_1 &= \frac{v(1 - 2\eta)}{1 - v} \\
&\quad - \frac{1}{2\gamma(1 - v)^2}\{v(1 - 2\eta) - \kappa(1 + 2v\eta)\}\{1 - \eta(3 - v^2) - \kappa v(1 - 2\eta)\} \\
T = M^1{}_2{}^1{}_2 &= \frac{1}{2}[1 - \eta(1 - v)]
\end{aligned}\right\} \tag{28a}$$

$$M^1{}_1{}^1{}_2 = M^2{}_2{}^1{}_2 = M^1{}_2{}^1{}_1 = M^1{}_2{}^2{}_2 = 0 \tag{28b}$$

where

$$\gamma = \bar{h} + \frac{1}{2(1 - v)}\{1 - 2v\kappa + \kappa^2 + \eta[-3 + 2v\kappa^2 + v^2 + \kappa(1 - v)(1 + 3v)]\}. \tag{28c}$$

Missing coefficients can in the chosen Cartesian coordinates only, be obtained by interchanging j and k, or m and n, respectively. Then the coefficients of the velocity field equations, eqn (11), may be expressed using eqn (12) and

$$\bar{L}^j{}_k{}^m{}_n = \frac{\rho E}{1 + v}\bar{M}^j{}_k{}^m{}_n \tag{29}$$

by means of the coefficients

$$\left.\begin{aligned}
\bar{M}^1{}_2{}^2{}_1 &= M^1{}_2{}^1{}_2 - (1 + v)\eta \\
\bar{M}^2{}_1{}^1{}_2 &= M^1{}_2{}^1{}_2 + (1 + v)\eta \\
\bar{M}^j{}_k{}^m{}_n &= M^j{}_k{}^m{}_n \quad \text{otherwise.}
\end{aligned}\right\} \tag{30}$$

4. APPLICATION TO STRAIN LOCALIZATION

The velocity field equations, eqn (11), could immediately be used to construct the point velocities, i.e. v^j at the actual configuration of any hypo-elastic body provided the coefficients $L^j{}_k{}^m{}_n$, or $\bar{L}^j{}_k{}^m{}_n$, respectively, are given. As a simple illustration we consider the case when they are constant, i.e. when pre-stress and pre-strain are homogeneous. If also body forces are disregarded then even the right-hand side of eqn (11) vanishes.

Now we examine first the uniaxial rod under longitudinal stress, the only relevant coefficient of which is given by eqn (21). Equation (11) assumes the shape

$$h\frac{\partial^2 v}{\partial x^2} = 0$$

where $v = v^1$ denotes the longitudinal velocity and x the longitudinal coordinate along the rod. For $h \neq 0$ the only solution v is a linear function in x so that homogeneous deformation continues. Bifurcation becomes possible as soon as

$$h = 0, \qquad \text{i.e.}\frac{\mathrm{d}Y}{\mathrm{d}\bar{\varepsilon}} = 0.$$

This result agrees completely with that of Hutchinson and Miles[6], in the limit of an infinitely slim rod of circular cross-section. It can for the uniaxial continuum, also be obtained directly by a classical stability consideration due to Considère[7].

In order to solve eqn (11) in general terms, it is advisable to reduce it to a first-order system by introducing the new variables

$$v^n{}_m = v^n|_m$$

(cf. Ref. [8]). They have to obey the analytical compatibility relations, i.e.

$$v^n{}_m|_l = v^n{}_l|_m = 0.$$

The independent ones of them form together with eqn (11), i.e.

$$\bar{L}^j{}_k{}^m{}_n v^n{}_m|_j = -\rho\dot{b}_k + (\rho g^m{}_n b_k - L^j{}_k{}^m{}_n|_j)v^n{}_m$$

the first-order system wanted. Again the right-hand side may for a homogeneously pre-stressed body, be put zero under statical conditions.

Now we proceed to the plate under uniaxial tension being quantitatively specified by eqns (29) and (30) in combination with eqns (26) and (28). Introducing matrix notation and observing the known symmetries as well as the fact that for Cartesian coordinates, upper and lower indices need not be distinguished we obtain

$$\mathbf{R}\,\mathbf{v}_{,1} + \mathbf{S}\,\mathbf{v}_{,2} = 0$$

where

$$\mathbf{v} = \begin{bmatrix} v^1{}_1 \\ v^1{}_2 \\ v^2{}_1 \\ v^2{}_2 \end{bmatrix} \quad R = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ P & 0 & 0 & U \\ 0 & T-(1+\nu)\eta & T & 0 \end{bmatrix} \quad \mathbf{S} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & .T & T+(1+\nu)\eta & 0 \\ W & 0 & 0 & 0 \end{bmatrix}.$$

After that the matrix

$$\mathbf{C}(\alpha) = \mathbf{R}\cos\alpha - \mathbf{S}\sin\alpha$$

has to be set up (cf. Ref. [5]). If there are real roots α of the characteristic equation, i.e.

$$
\begin{aligned}
-4\det(\mathbf{C}) = {} & [T(P+Q+U+W) + UW - PQ + (1+\nu)\eta(W-U) \\
& - (1+\nu)^2\eta^2]\cos^2 2\alpha - 2T(Q-P)\cos 2\alpha \\
& + [T(P+Q-U-W) - UW + PQ - (1+\nu)\eta(W-U) + (1+\nu)^2\eta^2] \\
= {} & 0
\end{aligned}
$$

then the problem is not elliptic, and α determines the direction of a characteristic at which, the highest spatial derivatives of the problem, i.e. $v^j{}_{k'l} = v^j{}_{,kl}$ may bifurcate. As this also could lead to a bifurcation of the solution as such, characteristics may be considered as possible deformation bands.

For metals the quantity, $\eta = Y/\rho E$ defining the "higher order" terms, is very small and may be disregarded. This yields

$$\cos 2\alpha = \frac{1-\kappa}{1+\kappa} \pm \frac{1}{1+\kappa}\sqrt{\left(\frac{-8\bar{h}}{1+\nu}\right)}$$

and can be interpreted as follows: α becomes real just at the maximum point of the generalized yield curve, i.e. $\bar{h} = h = 0$, and then the inclination of the deformation band amounts to

$$\cos 2\alpha = \frac{1-\kappa}{1+\kappa}. \tag{31}$$

The first condition, i.e. $h = 0$ again agrees under the conditions given, with that of a more direct, standard approach (cf. Refs [4, 9]) though this one examines bifurcations of lower order terms, i.e. $v^j|_k$ rather than bifurcations of $v^j|_{kl}$ as proposed here. Numerical calculations show that the critical value of h actually depends, on the "higher order" terms whose magnitude may be characterized by the quantity $Y/\rho E$ disregarded before. This is, at least for metals, of little practical significance only. Nevertheless that dependence may be different in this paper than in the literature mentioned because due to Section 3 the coefficients $L^k{}_j{}^m{}_n$ differ with respect to "higher order" terms.

The values of α corresponding to eqns (25) and (31) amount to

$\alpha = 45°$: shear band, Tresca

$\alpha = 0°$: necking, Tresca

$\alpha = 35.3°$: deformation band, Huber–v.-Mises.

The first two values are obvious, while the latter one agrees with a former analysis given by Thomas[11]. It appears too small as compared with experiments by Scholl (cf. Nádai and Wahl[10]). They show that for Lüders bands at the natural yield point of steel, $\alpha \approx 43°$ is a reliable average value.

5. CONCLUSION

Velocity field equations have been derived for the hypo-elastic body, and especially for the elastic–plastic body, with emphasis given to "higher order" terms. These equations govern the instantaneous point velocity field in the same way as, e.g. the Navier–Stokes equations do for viscous fluids. They are illustrated by applying them to bifurcation problems, of the thin rod and the thin plate both under uniaxial tension. The results agree fairly well with results known from the literature, obtained by direct lower order bifurcation analysis, or by stability considerations, respectively. Some discrepancies observed, might be due to the difference between the two approaches, and will be discussed in a future paper.

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